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Nonlinear thermomechanical analysis of laminated composite beams

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Abstract—A higher order beam finite element model for nonlinear static and dynamic thermoelastic response of laminated beams is presented. The model accounts for large deflections and nonuniform distribution of temperatures. Lateral strains are incorporated into the model by systematic reduction of two-dimensional laminate plate equations. A wide range of numerical examples are considered for nonlinear response of laminated beams subjected to thermal and thermomechanical loads. The influence of anisotropy, boundary conditions and stacking sequence on the thermally induced response is studied.

Keywords: laminated beam; thermal analysis; higher order theory; nonlinear analysis; finite element model.

1. INTRODUCTION

The increased use of fiber-reinforced composites in thermal environments has led us to investigate the thermal and thermomechanical analysis of laminated beams. A comprehensive review of literature indicates that most of the work is directed towards the analysis of heated laminated plates [1–5]. The results presented in references [2, 3] are based on classical laminated plate theory while those in references [4, 5] are based on first-order shear deformation theory.

Conventional beam models [6] neglect the effect of lateral strains. A constitutive relationship for laminated beams accounting for lateral strains has recently been reported [7]. It was shown in [7] that the application of conventional beam models results in a significant error in the analysis of angle-ply beams. In the present work, a finite element model is developed to study the nonlinear static and dynamic analysis of laminated beams subjected to thermal and thermomechanical loads. The formulation accounts for higher-order shear deformation effects and lateral strains. The

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von-Karman nonlinear strain–displacement relations are used to account for large thermal deflections.

2. MATHEMATICAL MODEL

A laminated composite beam is shown in Fig. 1. The laminate is made up of many unidirectional plies stacked in different orientations with respect to a reference axis. The length, breadth and height of the beam are represented by L, b and h, respectively, as shown in Fig. 1. A refined nonlinear laminated beam constitutive equation accounting for shear deformation and thermal effect is derived. The lateral strain is included in the formulation by a systematic reduction of two-dimensional plate constitutive equations, and the steps are as follows.

The stress–strain relations, accounting for transverse shear deformation and the thermal effect, in the plate coordinates for the kth layer, can be expressed as [5]

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x - \alpha_x T \\ \varepsilon_y - \alpha_y T \\ \gamma_{xy} - \alpha_{xy} T \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}_k, \qquad (1)$$

where Q_{ij} are the transformed elastic coefficient; $T = T(x, y, z, t)$ is the temperature distribution; and $(\alpha_x, \alpha_y, \alpha_{xy})$ are the thermal expansion coefficients in the plate coordinates, and are related to the coefficients $(\alpha_L, \alpha_T, 0)$ in the material principal directions.

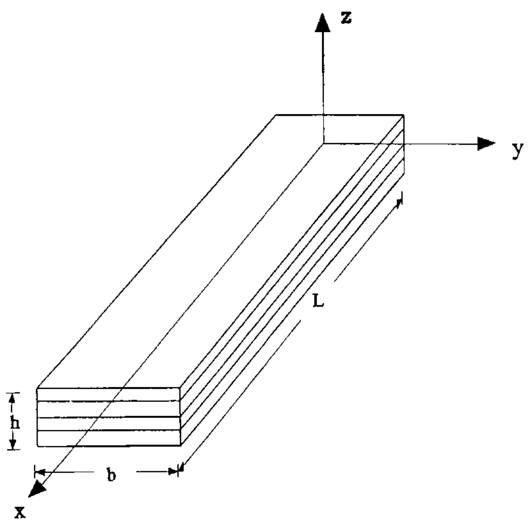

Figure 1. Geometry of a laminated beam.

For a laminated beam, the streses σ_y, τ_{xy}, τ_{yz} are assumed to be zero, while the lateral strains $\varepsilon_y - \alpha_y T$, $\gamma_{xy} - \alpha_{xy} T$, and γ_{yz} are retained. Following the procedure outlined in [7], equation (1) takes the form

$$\begin{Bmatrix} \sigma_x \\ \tau_{xz} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & 0 \\ 0 & \bar{Q}_{55} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x - \alpha_x T \\ \gamma_{xz} \end{Bmatrix}_k, \tag{2}$$

$$\bar{Q}_{11} = Q_{11} + \frac{Q_{16}Q_{26} - Q_{12}Q_{66}}{Q_{22}Q_{66} - Q_{26}^2} Q_{12} + \frac{Q_{12}Q_{26} - Q_{22}Q_{16}}{Q_{22}Q_{66} - Q_{26}^2} Q_{16}, \tag{3}$$

where

$$\bar{Q}_{55} = Q_{55} - \frac{Q_{45}^2}{Q_{44}}.$$

The strain–displacement relations based on a higher order shear deformation theory including the von-Karman strains are given by

$$\varepsilon_x = \varepsilon_x^0 + z\kappa_x^1 + z^3\kappa_x^2, \qquad \gamma_{xz} = \gamma_{xz}^0 + z^2\kappa_{xz}, \tag{4}$$

where

$$\begin{aligned} \varepsilon_x^0 &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \kappa_x^1 = \frac{\partial \phi}{\partial x}, \qquad \kappa_x^2 = -\frac{4}{3h^2}\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right), \\ \gamma_{xz}^0 &= \phi + \frac{\partial w}{\partial x}, \qquad \kappa_{xz} = -\frac{4}{h^2}\left(\phi + \frac{\partial w}{\partial x}\right), \end{aligned} \tag{5}$$

$u(x,t)$ and $w(x,t)$ in equations (5) are the midplane displacements, and $\phi(x,t)$ is the normal rotation. It may be noted that the transverse shear strain and transverse normal stress are zero at $z = \pm\frac{h}{2}$ (for more details, see [8]). This eliminates the need for the shear correction factor normally used in first-order shear deformation theory.

After substituting equations (5) and (4) in equation (2), the resulting equation is integrated through the thickness of the laminate. Then the laminated constitutive equations take the form

$$\begin{aligned} \begin{Bmatrix} N_x \\ M_x \\ P_x \end{Bmatrix} &= \begin{bmatrix} \bar{A}_{11} & \bar{B}_{11} & \bar{E}_{11} \\ \bar{B}_{11} & \bar{D}_{11} & \bar{F}_{11} \\ \bar{E}_{11} & \bar{F}_{11} & \bar{H}_{11} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \kappa_x^1 \\ \kappa_x^2 \end{Bmatrix} - \begin{Bmatrix} N_x^T \\ M_x^T \\ P_x^T \end{Bmatrix}, \\ \begin{Bmatrix} Q_{xz} \\ R_{xz} \end{Bmatrix} &= \begin{bmatrix} \bar{A}_{55} & \bar{D}_{55} \\ \bar{D}_{55} & \bar{F}_{55} \end{bmatrix} \begin{Bmatrix} \gamma_{xz}^0 \\ \kappa_{xz} \end{Bmatrix}, \end{aligned} \tag{6}$$

where

$$\{N_x, M_x, P_x\} = \int_{-h/2}^{h/2} \sigma_x(1, z, z^3)\,\mathrm{d}z, \qquad \{Q_{xz}, R_{xz}\} = \int_{-h/2}^{h/2} \tau_{xz}(1, z^2)\,\mathrm{d}z,$$

$$\{N_x^T, M_x^T, P_x^T\} = \int_{-h/2}^{h/2} \bar{Q}_{11}^k \alpha_x(1, z, z^3)T\,\mathrm{d}z,$$

$$\{\bar{A}_{11}, \bar{B}_{11}, \bar{D}_{11}, \bar{E}_{11}, \bar{F}_{11}, \bar{H}_{11}\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} (1, z, z^2, z^3, z^4, z^6)\,\mathrm{d}z, \tag{7}$$

$$\{\bar{A}_{55}, \bar{D}_{55}, \bar{H}_{55}\} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \bar{Q}_{55}^k(1, z^2, z^4)\,\mathrm{d}z,$$

and n denotes the number of layers.

The temperature distribution in the bean can be expressed as

$$T(x, z, t) = T_1(x, t) + \frac{z}{h} T_2(x, t). \tag{8}$$

3. FINITE ELEMENT FORMULATION

The mathematical statement of Hamilton's principle is expressed here as

$$\int_{t_1}^{t_2} \delta(T - U + W)\,\mathrm{d}t = 0, \tag{9}$$

where T is the kinetic energy, U is the strain energy and W is the work done by external forces.

The variation in kinetic energy of the beam can be written as

$$\delta T = \int_l \{\delta\dot{u}\}^T [\bar{M}]\{\dot{u}\} b\,\mathrm{d}x, \tag{10}$$

where

$$\{u\} = \left\{u, w, \frac{\mathrm{d}w}{\mathrm{d}x}, \phi\right\}^T, \tag{11}$$

and

$$[\bar{M}] = \begin{bmatrix} I_1 & 0 & 0 & 0 \\ 0 & I_1 & 0 & 0 \\ 0 & 0 & I_7 & \overline{I}_1 \\ 0 & 0 & \overline{I}_1 & \overline{I}_2 \end{bmatrix}, \tag{12}$$

and

$$\bar{I}_1 = \frac{16}{9h^4} I_7 - \frac{4}{3h^2} I_5,$$
$$\bar{I}_2 = I_3 + \frac{16}{9h^4} I_7 - \frac{8}{3h^2} I_5, \tag{13}$$
$$(I_1, I_3, I_5, I_7) = \int_{-h/2}^{h/2} \rho(1, z^2, z^4, z^6)\, dz.$$

The variation in strain energy can be expressed as

$$\delta U = \int_l \{\delta\varepsilon\}^T \{N\} b \, dx, \tag{14}$$

where

$$\{N\} = \left\{N_x, M_x, P_x, Q_{xz}, R_{xz}\right\}^T,$$
$$\{\varepsilon\} = \left\{\varepsilon_x^0, \kappa_x^1, \kappa_x^2, \gamma_{xz}^0, \kappa_{xz}\right\}^T. \tag{15}$$

Finally, the virtual work done by external forces is given by

$$\delta W = \int_l p \delta w b \, dx, \tag{16}$$

where $p(x, t)$ is a uniformly distributed load.

The generalized displacements u, w and ϕ are approximated by

$$u(x, t) = \sum_{j=1}^{2} u_j(t) N_j(x),$$
$$w(x, t) = \sum_{j=1}^{4} w_j(t) \psi_j(x), \tag{17}$$
$$\phi(x, t) = \sum_{j=1}^{2} \phi_j(t) N_j(x),$$

where $N_j(x)$ are the linear Lagrange interpolation functions, and $\psi_j(x)$ are the Hermite cubic interpolation functions.

Using equations (10), (14), (16), (17) and (9), the following finite element equations are obtained:

$$[K]^e\{\Delta\}^e + [M]^e\{\ddot{\Delta}\}^e = \{F\}^e, \tag{18}$$

where $[K]^e$ is the nonlinear stiffness matrix, $[M]^e$ is the element mass matrix and $\{F\}^e$ is the element force vector which includes forces due to thermal and mechanical loads.

4. SOLUTION PROCEDURE

The element equations (18) are assembled, boundary conditions are imposed and the resulting equations are solved using the Newmark constant acceleration scheme in conjunction with a direct iteration scheme. Using Newmark's method, equation (18) can be written as

$$[\widehat{K}]\{\Delta\}_{n+1} = \{\widehat{F}\}_{n,n+1}, \qquad [\widehat{K}] = [K] + a_0[M], \tag{19}$$

where

$$[\widehat{F}] = \{F\} + [M]\big(a_0\{\Delta\}_n + a_1\{\dot{\Delta}\}_n + a_2\{\ddot{\Delta}\}_n\big), \qquad a_0 = \frac{1}{\beta\Delta t^2}, \quad a_1 = a_0\Delta t, \quad a_2 = \frac{1}{2\beta} - 1. \tag{20}$$

Once the solution $\{\Delta\}$ is known at $t_{n+1} = (n+1)\Delta t$, the first and second derivatives (velocities and accelerations) of $\{\Delta\}$ at t_{n+1} can be computed from

$$\begin{aligned} \{\ddot{\Delta}\}_{n+1} &= a_0\big(\{\Delta\}_{n+1} - \{\Delta\}_n\big) - a_1\{\dot{\Delta}\}_n - a_2\{\ddot{\Delta}\}_n, \\ \{\dot{\Delta}\}_{n+1} &= \{\dot{\Delta}\}_n + a_3\{\ddot{\Delta}\}_n + a_4\{\ddot{\Delta}\}_{n+1}, \end{aligned} \tag{21}$$

where $a_3 = (1-\alpha)\Delta t$ and $a_4 = \alpha\Delta t$. The parameters α and β are taken to be 0.5 and 0.25, respectively.

Using direct iteration technique, equation (19) can be expressed as

$$[\widehat{K}\{\Delta^r\}_{n+1}]\{\Delta^{r+1}\}_{n+1} = \{\widehat{F}\}_{n,n+1}, \tag{22}$$

where $\{\Delta^r\}$ denotes the solution vector obtained in the rth iteration. At the end of each iteration, the solution is checked for convergence. The convergence criterion is given by

$$\left[\sum_i^m |\Delta_i^r - \Delta^{r+1}|^2 \Big/ \sum_i^m |\Delta_i^r|^2\right]^{1/2} \leqslant 0.01,$$

where m denotes the total number of nodes in the mesh.

5. RESULTS AND DISCUSSION

A wide variety of numerical examples is discussed to illustrate the nonlinear static and dynamic behavior of laminated beams subjected to thermomechanical loadings. For the sake of brevity, a uniform temperature distribution ($T = T_1$), a temperature distribution that varies linearly through the thickness ($T = (z/h)T_2$), and a combination of both are considered. Unless stated, computations are carried out using 10 beam elements with a length to thickness ratio of 50 ($L = 0.254$ m). The boundary conditions used in the analysis are shown in Table 1.

The accuracy of the present beam model is first demonstrated by comparison to the results for a linear static analysis. Table 2 shows the comparison of nondimensional maximum deflections ($\overline{w}$) of SS thin laminated beams under a uniformly distributed mechanical load (p). The results are compared with the exact solutions obtained using a conventional beam model [6]. It is observed that for orthotropic and cross-ply beams there is hardly any noticeable different, but there exists a considerable difference in the case of angle-ply laminates. This is due to the fact that the laminated beam theory considered in reference [6] neglects the lateral strain.

Having validated the present finite element model, a number of laminated beam problems under thermal loadings are considered using the basic ply properties of graphite/aluminum given in Table 3.

Table 1.
Boundary conditions

Type	$x = 0$	$x = L$
Clamped–Free (CF)	$u = w = \phi = 0$	–
Clamped–Clamped (CC)	$u = w = \phi = 0$	$u = w = \phi = 0$
Clamped–Supported (CS)	$u = w = \phi = 0$	$u = w = 0$
Simply-Supported (SS)	$u = w = 0$	$u = w = 0$

Table 2.
Comparison of nondimensional center deflections ($\overline{w}$) of SS laminated beams; $E_1 = 144.8$ GPa, $E_2 = 9.65$ GPa, $G_{12} = G_{13} = 4.14$ GPa, $G_{23} = 3.45$ GPa, $\nu_{12} = 0.3$, $\overline{w} = w\left(\frac{E_2 I}{pL^4}\right) \times 10^2$, $I = \frac{bh^3}{12}$, $\frac{L}{h} = 100$

	$\overline{w}$	
Lamination scheme	Conventional beam model [6]	Present beam model
0°	0.08631	0.08153
[0°/90°/90°/0°]	0.09771	0.09224
[45°/ − 45°/ − 45°/45°]	0.28291	0.94663
[30°/50°/50°/30°]	0.15696	0.62281

Table 3.
Material properties of graphite/aluminum

E_1 (N/m^2)	190.0×10^9
E_2 (N/m^2)	48.3×10^9
G_{12} (N/m^2)	17.3×10^9
ν_{12}	0.28
G_{13} (N/m^2)	16.5×10^9
G_{23} (N/m^2)	16.5×10^9
ρ (kg/m^3)	2400.0
α_1 (m/m/°C)	3.34×10^{-6}
α_2 (m/m/°C)	26.1×10^{-6}

5.1. Nonlinear static analysis

Figure 2 shows the linear and nonlinear center deflection of a [0°/90°/90°/0°] simply supported cross-ply beam subjected to a linearly varying temperature field. Figure 3 shows the effect of boundary conditions on the nonlinear center deflection. The effect of L/h ratio on the nonlinear center deflection of a simply-supported cross-ply laminate is shown in Fig. 4. It is evident from Fig. 4 that the nonlinear center deflection increases with an increase in L/h ratio.

Figure 5 shows the effect of ply orientation on the nonlinear center deflection of a simply-supported laminated beam. It is observed that the nonlinear center deflection in the [0°/90°/0°/...] cross-ply laminated beam is much smaller because it is stiffer when compared to an angle-ply beam. The effect of number of layers on the nonlinear center deflection of a simply-supported cross-ply beam subjected to a linearly varying temperature field is shown in Fig. 6. It can be seen from Fig. 6 that the nonlinear center deflection decreases with an increase in the number of layers. This is because of a decrease in the coupling effect between bending and extension with an increase in the number of layers.

5.2. Nonlinear dynamic analysis

Figure 7 shows the linear and nonlinear dynamic response of a simply-supported [0°/90°/90°/0°] cross-ply beam subjected to a combined uniform and linearly varying temperature distribution ($T = T_1 + (z/h)T_2$). The effects of ply orientation and L/h ratio on the nonlinear dynamic response are shown in Fig. 8 and Fig. 9, respectively. Figure 10 shows the effect of number of layers on the transient response of a simply-supported cross-ply laminated beam. It is seen from Fig. 10 that the amplitude of the response decreases with an increase in the number of layers. Finally, the transient response of a simply-supported cross-ply laminated beam for different types of loadings is studied in Fig. 11.

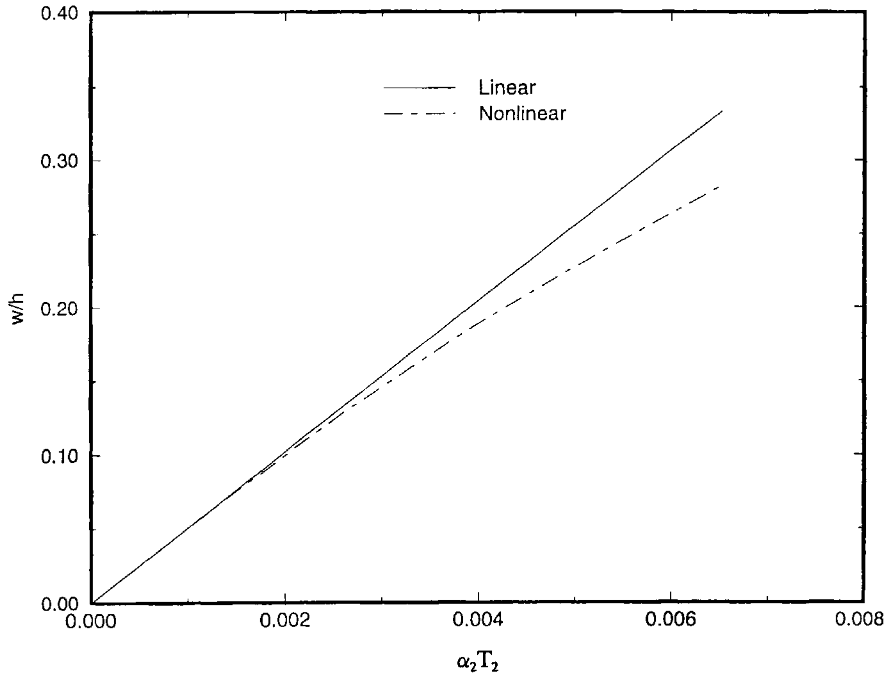

Figure 2. Linear and nonlinear center deflection of a simply-supported [0°/90°/90°/0°] beam.

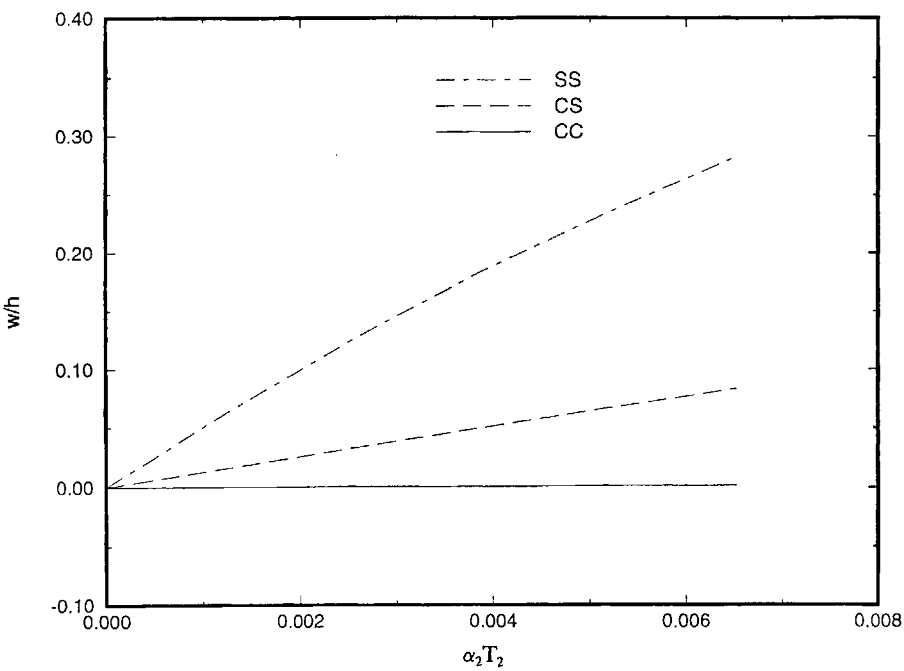

Figure 3. Effect of boundary conditions on nonlinear center deflection of a [0°/90°/90°/0°] beam.

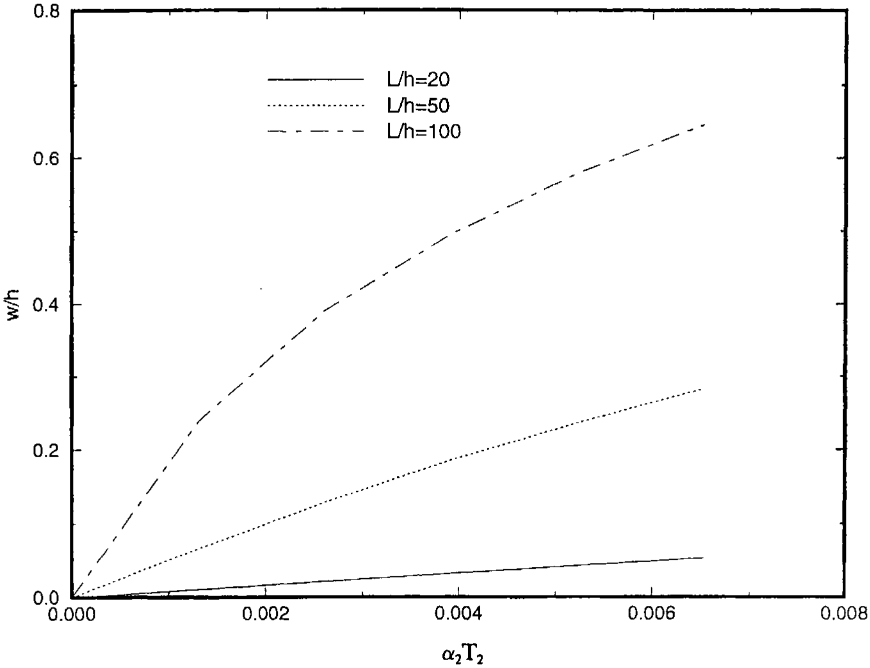

Figure 4. Effect of L/h ratio on nonlinear center deflection of a simply-supported [0°/90°/90°/0°] beam.

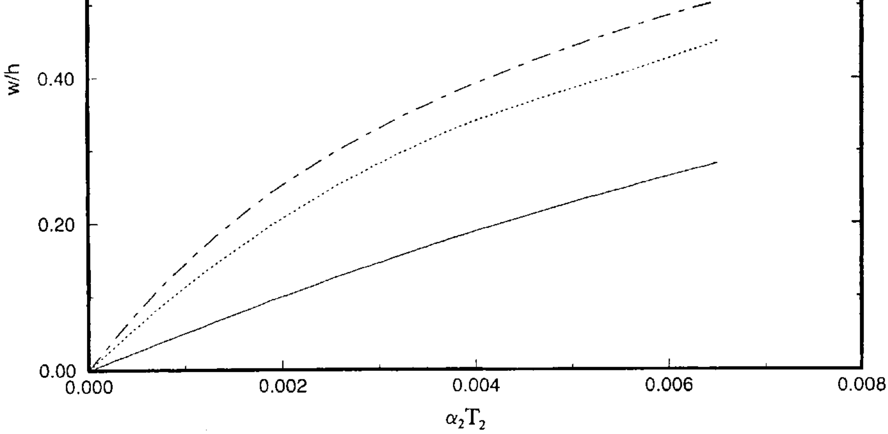

Figure 5. Effect of ply orientation on nonlinear center deflection of a simply-supported beam.

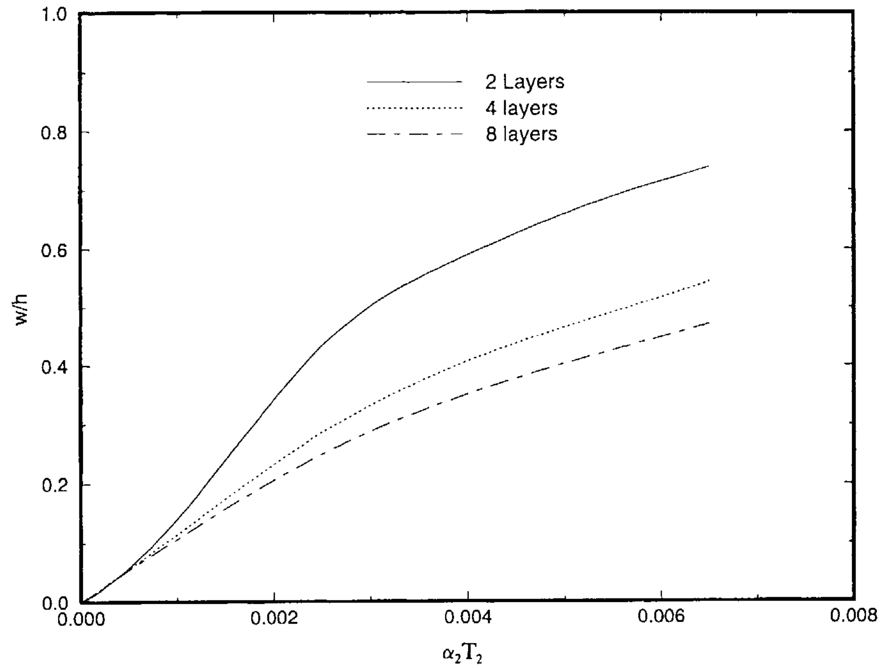

Figure 6. Effect of number of layers on nonlinear center deflection of a simply-supported [0°/90°/90°/0°] beam.

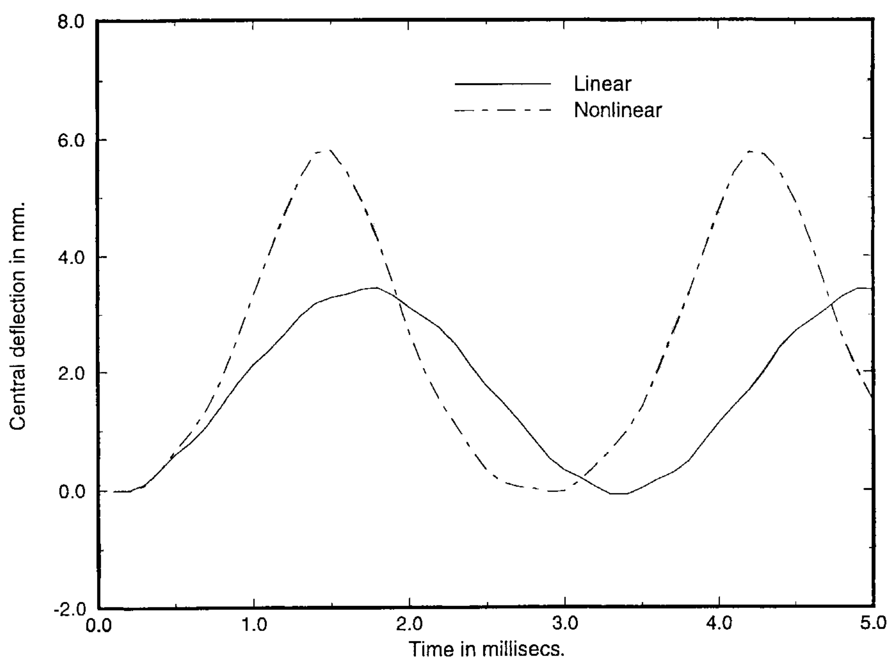

Figure 7. Linear and nonlinear transient response of a simply-supported [0°/90°/90°/0°] beam (T_1 = 100°C, $T_2 = 250$°C).

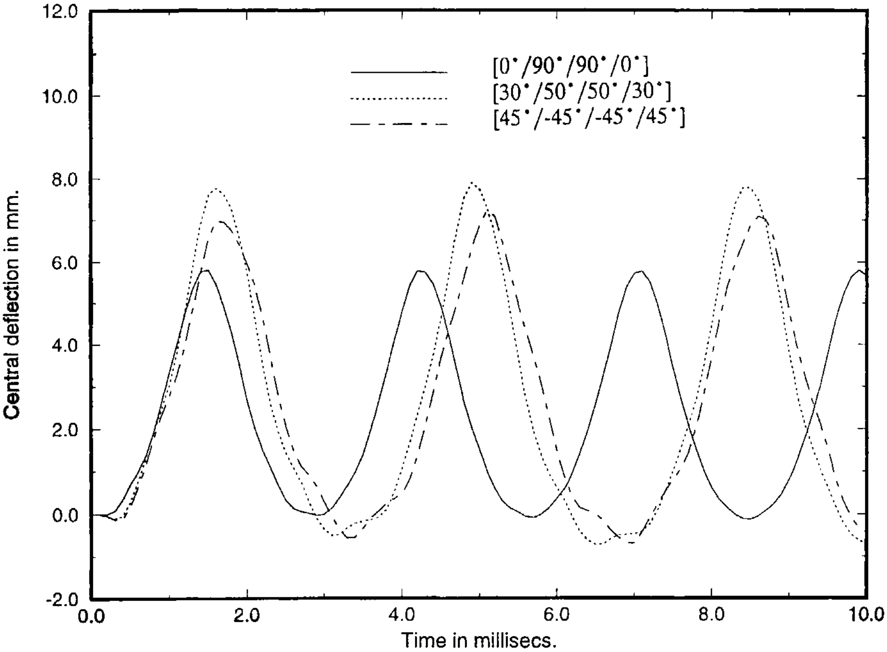

Figure 8. Effect of ply orientation on nonlinear transient ($T_1 = 100$°C, $T_2 = 250$°C).

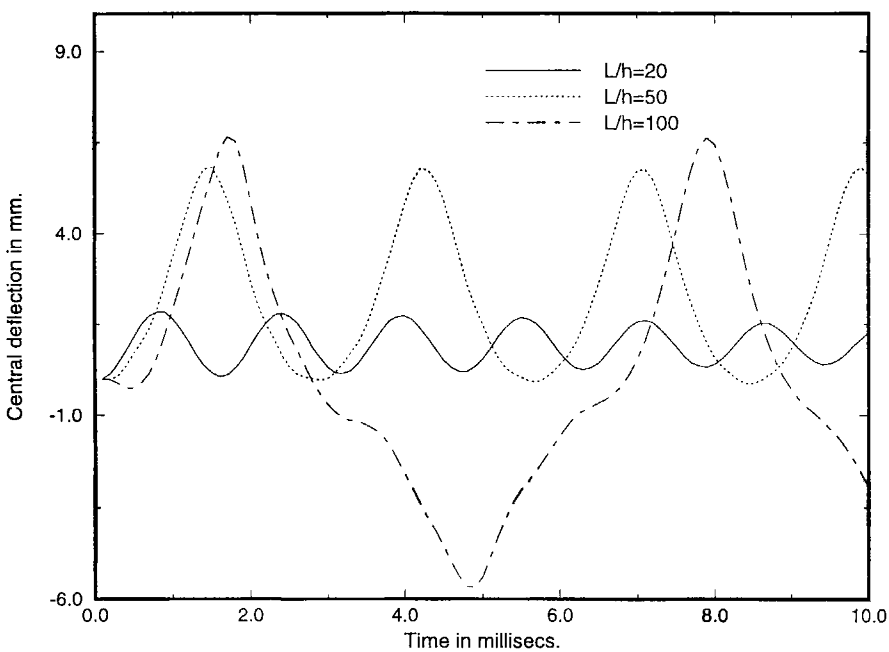

Figure 9. Effect of L/h ratio on nonlinear transient response ($T_1 = 100$°C, $T_2 = 250$°C).

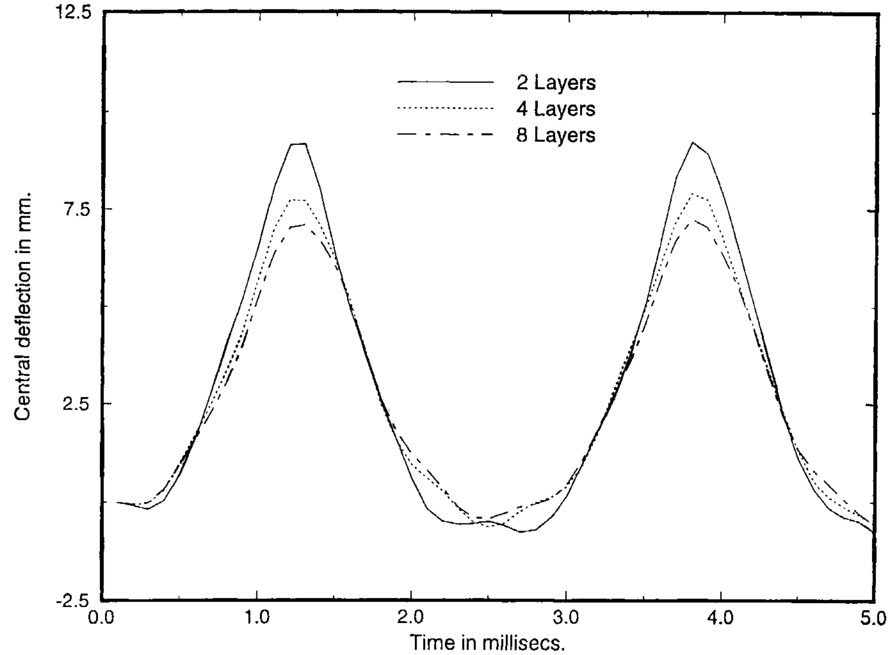

Figure 10. Effect of number of layers on nonlinear transient response ($T_1 = 100°C$, $T_2 = 250°C$).

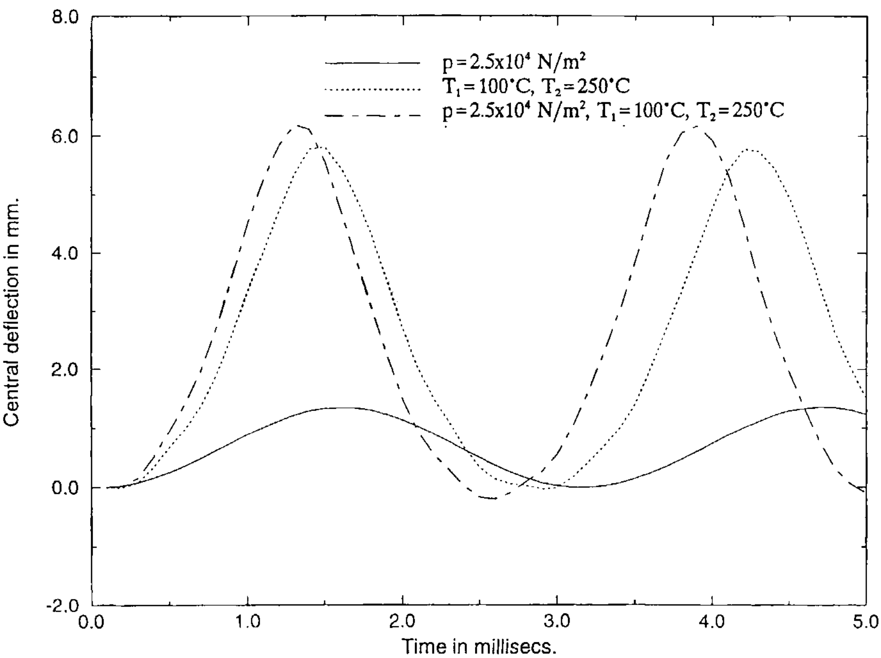

Figure 11. Nonlinear transient response of a simply-supported [0°/90°/90°/0°] beam subjected to combined loadings.

6. CONCLUSIONS

A one-dimensional finite element formulation based on a higher order theory is developed for the thermal and thermomechanical analysis of laminated beams. The model accounts for geometric nonlinearity and lateral strains. Also, the model accounts for all couplings that characterize generally laminated beams. Numerical results are presented for the static and dynamic thermal responses of laminated beams.

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